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$\mathcal{A}$-invariance: An Axiomatic Approach to Quantum Relativity

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Abstract The sort of approach claimed by the title of this article is realizable, at least, within the *framework of ADG* where we do not assume any "*spacetime*" supplying the dynamics we employ. The latter classical type of argument can naturally be included herewith along with its concomitant impediments that are emanated therefrom and are essentially "*absorbed*", technically speaking, *by the proposed mechanism.* So our approach, being "*manifoldless*" (thence, *no smoothness*, in the standard sense) does not contain any such issue, as before, according to the very definitions, being thus "*singularities*"-*free*. As a consequence, the equations that one would be able to formulate within the present set-up will be, by the very essence of the matter, already the *quantized* ones.

Keywords Quantum relativity · Gel'fand/Yoneda transform · "Singularities" · Sheaf-theoretic dynamics · Dynamical relativistic localization · Yang-Mills field · Topos-theoretic dynamics · Topos-theoretic quantum field theory · Topological algebra space · Topological algebra scheme

1 $\mathcal{A}$-invariance

It is well understood that we can isolate/detect something, only, if this is "$\mathcal{A}$-*invariant*". Here $\mathcal{A}$ denotes our "*arithmetic*" we use each time. In particular, we posit that

(1.1) *a physical law is* $\mathcal{A}$*-invariant*.

See also (1.4) in the sequel. Of course, any *physical law is* in effect *independent of us*. Therefore, it is reasonable to expect that its realization/detection should not depend on the

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"arithmetic", *we* employ at each particular time to detect it. Thus, one can actually supplement (1.1) saying that

(1.2) *a physical law is always $\mathcal{A}$-invariant, for any $\mathcal{A}$ whatsoever*. Yet, we can further note that

(1.2.1) *a physical law penetrates everything* since it is actually there always, and the same.

In this context, we can also remark that the above still points out an essentially *abuse of language* we usually employ any time we refer to (1.1) in the sense that,

(1.3) it is actually *we* who at each particular instance concoct our "*arithmetic*" $\mathcal{A}$, so that the physical law at issue be "*$\mathcal{A}$-invariant*"; hence *detectable* (: it is essentially only then (cf. (1.1) that one can perceive the law the same being in effect independent of us). Therefore, *applicable* then, as well; see e.g. *general relativity*, along with (1.17) in the sequel.

So to say it once more, equivalently,

(1.3′) it is *we* who have succeeded in getting a "*functorial*" way/"*calculus*" of expressing what we are looking for; hence of being able to detect a *physical law, something always functorial*, that is "*$\mathcal{A}$-invariant*" *for any $\mathcal{A}$ whatsoever* (cf. thus (1.2)).

On the other hand, it is still instructive that our nowadays *perception of Physis* allows us to say that

(1.4) "*physical geometry*" (cf. *P. Bergman* [1]) is the outcome of the physical laws.

See for instance *A. Mallios* [19, (1.1)]. Hence as a spin-off of (1.2) and (1.4) we actually conclude that:

(1.5) *The Physis* herself *is $\mathcal{A}$-invariant for any $\mathcal{A}$ whatsoever*. In other words, we might equivalently say that

(1.5.1) *Nature is "functorial"*,

in the way we understand it. See also (1.3) above, along with (1.8) in the sequel. Thus, what we can further say here is that;

(1.5.2) in view of (1.5.1) *we should behave*, i.e., make our calculations, accordingly, viz. *in a "functorial way"* as well, if we want to detect a physical law.

Consequently,

(1.6) *we should* always *choose* such an $\mathcal{A}$ *that* (1.5), hence, also (1.2), *be true*. See also the Examples below.

In conclusion, one might further look at the preceding as

(1.7) another equivalent formulation of the classical "*Principle of General Covariance*" (see, for instance, *R. Torretti* [38, p. 153]). Yet, *equivalently*, of the "*Principle of General Relativity*" or even of the so-called "*Gauge Principle*" (see e.g. *M. Nakahara* [30, p. 28 and 10, respectively]).

We can thus realize here that *an* old *axiom is* now simply *made*, according to ADG, *into a theorem.* Hence, one obtains another remarkable *potential intervention/application*, in effect of the point of view *of* ADG on the classical perspective.

Examples 1.1 (i) Within the previous context we remark that (1.5) is actually realized by the same *operation of* doing "*geometry*" *already from its inception* (: "measure"—$\mu\varepsilon\tau\rho\widetilde{\omega}$ (metrō)) [Greek]). The ensuing case is still more enlightening.

(ii) For the terminology applied in the sequel we refer to *A. Mallios* [15, 20]: Now, by looking at the notion of "$\mathcal{A}$-*connection*" ($\longleftrightarrow$ physical law $\longleftrightarrow$ field), ibid., see also *A. Mallios* [21], we remark that; we usually apply an $\mathcal{A}$-*connection* which is "($\mathcal{A}$-)*metric invariant*", or else, an ($\mathcal{A}$-)*connection compatible with the* ($\mathcal{A}$-)*metric* employed. However here as anywhere else, it is *we* again who have chosen (we should, in effect, see thus (1.6) above), such an $\mathcal{A}$-*metric* that *the law* (: $\mathcal{A}$-*connection*) *be* $\mathcal{A}$-*invariant* (true, its "*curvature*"), therefore *detectable* as well. Cf. also the following.

> *Note 1.1* It is worth remarking here that the aforementioned *choice of* $\mathcal{A}$ depends, as a matter of fact, on the *level of our theory* concerning the manner *we afford to describe Nature*. Yet, this same *choice of* $\mathcal{A}$ as above makes us in effect *more sensible/"real", effective* in the way of participating Nature, see also *A. Mallios* [21, (3.7)].

In toto, what we conclude is that

(1.8) we can locate *only* something which is "*compatible*" (: "*invariant*") *with* our own way of "*measuring*" (: collecting "*information*" about) *what we perceive/*observe.

The aforesaid "*compatibility*" that is the "*invariance of the result*" we get *by* "*measuring*" (: when, namely, following a (physical) procedure) is what we actually understand by an

$$\mathcal{A}\text{-}(co)variance \tag{1.9}$$

or even an

$$\mathcal{A}\text{-}(syn)variance \tag{1.10}$$

(*I. Raptis*), see also *A. Mallios–I. Raptis* [23]. It is the same situation that leads us on the basis of the preceding to characterize/realize something as a particular *event.* Yet, the same can of course be directly connected with the "*invariance*" of measurements with respect to (inner) transformations, or even "*change of coordinates*" (: "*representations*"), thus "*gauge*" of the "*arithmetic*" applied. In this connection see also, for instance, *I.R. Shafarevich* [35, p. 160ff].

On the other hand, by further employing herewith *technical language* the above may still be described as supplying an

$$\mathcal{A}\text{-}tensor \tag{1.11}$$

or, even simply a "*tensor*", when $\mathcal{A}$ is easily understood from the context. Therefore, the same thing as in (1.11) is *detectable*/perceived by us. It is still *worth remarking* the *omnipresence* of "$\mathcal{A}$" and its function throughout the preceding. Of course,

(1.12) "$\mathcal{A}$" *means* actually *we*,

see also (1.3); thus,

(1.13) it is certainly *important* any time *we* can "*identify*" $\mathcal{A}$ (: *we*) *with* what we observe (: *Physis*). This especially, *without any subsidiary means* through which we can perform our function (: "identification", as for instance, by resorting as usual to the notion of what we perceive as "*spacetime*").

In this connection see also Sect. 3 below. Yet and this is also *of importance, the manner we perform the* aforementioned

(1.14) "*identification*" *of* $\mathcal{A}$ (: we, the observers) *with Physis* (: what we observe) is such that we can also follow/participate the observed "*variation*" *of* the "*eventsr*". The latter situation is characterized as the "*dynamics*" of the Physis (: the world we observe). Of course, a further immediate implementation of the previous (physical) function is the (so-called) "*kinematics*". We have actually here two *issues* that classically are *expressed through* our *Calculus*. See however below (: ADG).

The way the above (physical) function can be conceived, viz. *our function of participating*/following *the* (physical) "*dynamics*"/*variation* is usually achieved, technically speaking, through our (Newtonian-Cartesian) *Calculus*, i.e., by the type of "*differentiation*" we afford, which is still essentially referred directly to $\mathcal{A}$, as well. Therefore, we can say that

(1.15) *we do*, in effect, "*differentiate*" (either consciously, or even technically/mathematically speaking, by concocting "*Calculus*"),

in order *to take into account* (: participate/detect) *the* effectuated *variation of the events*. In this context, see also *S. C. Chern* [2] when quoting *C. H. Taubes* (ibid., p. 681).

In toto, we can further say that,

(1.16) one can locate *only something*, which is "*compatible" with* our own manner of "*measuring* (: detecting/collecting "*information*" about *what we perceive*). That is, in other words, *what is $\mathcal{A}$-invariant* in the sense we consider it in the preceding. Of course, we still have in mind here the following technical, yet important, relation by the very definitions,

(1.16.1) $$\mathbb{C} \underset{\rightarrow_{\varepsilon}}{\subset} \mathcal{A},$$

(see also (4.16) in the sequel and subsequent comments therein).

Furthermore, what we observe/perceive is certainly expressed, in technical terms, through our own "*arithmetic*" (theory) $\mathcal{A}$ we afford at each particular period of time/evolution of our science. Thus the previous *relation* (1.16.1) can also be construed simply, just, as *a particular instance of the* aforesaid *evolution* depending on the *type of $\mathcal{A}$ we provide*.

On the other hand, within the same vein of ideas, one may still support that

(1.17) "*relativity*" in general, means the (physical) realization/effectuation of following/detecting a "*general covariance*", with respect to our own "*arithmetic*" $\mathcal{A}$, any time we refer to physical fields (: laws).

Furthermore, within the same context we also remark that in the case of *classical* (: in terms of CDG) *General Relativity*, we usually say that

(1.18) the *geometrical relations* (: equations) defined on a 4-dimensional (smooth) manifold X, which turn it into [what we call] a "*spacetime*" *become* "*variable*".

However, meditating now within the context of ADG we realize in effect that:

(1.19) *what actually becomes* "*variable*" (i.e., it is *registered* as such), *is* not, just, X (the 4-dimensional manifold of CDG), which, as it were, does not even exist according to ADG, in the sense at least we mean it in the classical theory, but *our* "*antenna*", that is *the* "*sheaf of coefficients*" $\mathcal{A}$. Thus in other words, "*we*", the observers, who also make "*calculations*", yet write down equations (needing thereby a "*canvas*"); something, more important, we further note here that this is made (reminding thus us "$\mathcal{A}$-*invariance*") in such a manner that

(1.19.1) *the form of the* latter *equations is independent of the* aforesaid *variation*.

Therefore, in other words,

we are thus in a position to work,

(1.20) *in a* so to say "*functorial way*",

hence, *more* "*physically*" (see also below).

So, as a consequence of the previous remarks, we come once more to the conclusion that,

(1.21) whenever we speak of "*spacetime*" we are actually referred, not to our (physical) "*environment*", *but* just *to* our *technique of* "*measuring*", yet "*calculating*", hence writing also down "*equations*", therefore *at the very end, to a* certain *particular* $\mathcal{A}$, the choice of the latter depending (as this should actually happen...) on the particular case at issue.

Concerning the terminating remark of the above, see also for instance [16, 18, 22–24, 26, 31, 41]. Thus, we are led still herewith a posteriori to a known relevant utterance of A. Einstein, see (3.12.2) below.

What we can further note, in conjunction with the preceding and based also on the very nature of ADG and its physical applications, pertaining in particular to the so-called "*singularities*" (see e.g. *A. Mallios–E.E. Rosinger* [24, 25]), is the following.

(1.22) Our endeavour to be "$\mathcal{A}$-*covariant*", as above, entails also that we may/(should actually) disregard the classical *principle of uncertainty* (Heisenberg) and/or *principle of complementarity* (Bohr), provided we work within ADG. Indeed, "*Physis has no "singularities"*" (see *A. Mallios* [16, (4.4), (4.5) and (6.4), (7.9), along with (10.10), (10.11)]).

We are going now to discuss by the following Sections the way the previous account might still be conceived in *categorical terms* by isolating/illuminating the *esoteric character of the means* through which *we* usually *perceive the* aforesaid "*variation*" (: "*dynamics*") or even "*differentiation*". In this context, we may still remark here, by paraphrasing *E. Galois* that, (1.23) it is *not* always "*the calculations*" that are thus much *of importance*, as the conclusions (: physical significance) which one can arrive at, when trying to understand the *context* of the outcome *of these calculations*. This task is so better supported as the same (calculations) are of a more "*categorical*" (: "*geometrical*", hence, physical) nature. See also the adage of the same *Evariste Galois*: "*Les calculs sont impraticables*" (emphasis ours).

2 Differentiation/Dynamics

What we perceive in the previous Section as our "*arithmetic*" and what we usually employ up to these days may be characterized, as an *analytical* procedure in the broad sense of term yet, as an *algebraic* one (*Warning*, *not* "*categorical/functorial algebraic*", see below), or even a *numerical*/analytic one: Of course, this refers to an "*arithmetic, based on* the notion of *numbers*, either in the *classical* sense of the latter term, viz. by considering types of *numerical algebraic systems*, e.g. Euclidean (finite dimensional vector) spaces, which are either *globally* (: *affinely*), or even *locally* (: "*manifold*"*-like*) *conceived*, or yet otherwise via an *abstract algebraic system* the elements of which might thus be construed as "*generalized numbers*". Now, the latter type of "*numbers*" are usually effectuated, through their conversion into (abstract) *functions*, indeed in the broadest sense of this term, as for instance *sections*, functors, and the like (this especially lately), by means of appropriate *representation* procedures, e.g., *Gel'fand* or even *Yoneda transforms* (see below).

The preceding can also be viewed as describing the type of *analysis/arithmetic* that we actually employ, up to the present time, by majority if not even exclusively. The same sort of analysis as above might also be characterized by simply looking at historical facts as a

(2.1) *Newtonian–Cartesian arithmetic/analysis.*

On the other hand, by paraphrasing for instance herewith *D. R. Finkelstein* (see [8, p. 155], we can further remark that

(2.2) the above *Newtonian*(-Cartesian) *character of* our *analysis* has dominated, undebatably, our perception/description of the Nature, until the advent of quantum theory. Then, things started to be changed, thus numbers appear to be more "*functional*", as e.g. first matrices (*Heisenberg*), then appropriate "operators" (*von Neumann*).

In this connection the contribution of general relativity, as it concerns the situation described by (2.1), that is the application of classical *differential geometry of* smooth *manifolds* (CDG), has been certainly instrumental; however, an opposite critique to the latter framework due exactly to the appearance of quantum (field) theory was already made by *Einstein* himself even in the early days of general relativity (1920), as well as, near his later days by suggesting that there was to be found,

(2.3) "*... a purely algebraic theory for the description of reality.*"

[5, p. 166] (emphasis above is ours).

In this context one can still remark here that *Einstein* did not consider *differential geometry* (: CDG, anyway at that time), *as an end* pertaining to general relativity, in the sense that one should convert the latter theory within the framework of the former, alias in the "*geometrization*" of the same theory; instead he was always looking, simply to paraphrase here *P.G. Bergman* [1] at

(2.4) … fusing [any] *mathematical structures to represent physical fields*.

Thus *Einstein* was really annoyed at realizing that

(2.5) the inherent in CDG "*local analysis*" (based of course by the very definitions, on a *locally euclidean/Newtonian* framework) was in effect always responsible for the appearance of the ever pestilential "*singularities*", viz. *infinities*, "*anomalies*" and the like.

Yet, this when trying to fuse/combine general relativity with quantum theory, alias to have *his equations in a "quantized form*". Thus he was led even to ask for *expelling the continuity* (: *limits*) from the calculations; limits appear of course always in the *Newtonian*/"*geometrical*" aspect of the derivative. Cf. here for instance *A. Mallios* [16], as well as (2.3) above. In this regard, it is still worthwhile to recall at this point relevant thoughts of *Dirac* [3, p. 85], *Feynman* [9, p. 166] and *Isham* [11, p. 393] (see also *A. Mallios* [18, (1.1′), (1.3) and (1.4)]).

So in connection with (2.5), we also remark as a general moral of what we perceive, through ADG, that the so-called, classically speaking (see also [4], along with [21, (3.2)]),

(2.6) "*singularities*", are just a mathematical description/ expression of the situation we are confronted with when dealing with the "*quantum deep*", using of course the classical (: *Newtonian-Cartesian*) $\mathcal{A}$.

Now the same *algebraic/numerical arithmetic* could be *relativized*, viz. converted into a "*variational*" one by simply employing herewith *sheaf-theoretical* methods/techniques, *not fiber bundle theory* (: principal/vector bundles and the like). See, for instance, *A. Mallios* [17], *R. Haag* [10]. So it is also recently realized that

(2.7) "… *instead of a fiber bundle one has to work with a sheaf*".

[10, p. 326]. Indeed, one further remarks with [10] that

(2.8) "… *quantities associated with a point are very singular objects* [so] it is *advisable to consider neighborhoods*".

Hence, the choice of a *sheaf* as in (2.7). Yet, one can refer here to the already classical considerations of *R.D. Sorkin* [36], in substituting the standard *"pointwise/defined*" (: "continuous") *topology*, by a "*finitary*" (viz. via open coverings determined) one, as well as to relevant applications of *I. Raptis* [33], along with further work of his (see e.g. [34]). In toto, one concludes that the

(2.9) *applications of sheaf theory* to confront with issues of *general relativity* proves to be particularly effective, as well as, more *natural*, while the same framework is also *adaptable* by its very nature *to localization* arguments; the latter are still of a special significance, pertaining to relevant questions connected with *quantum field theory*, *Haag* (ibid.), hence in particular with *quantum relativity*, as well.

Therefore, we can also sum up by saying that

(2.10) *relativize/vary* fits in, most appropriately, with a suitably of course chosen *sheaf–theoretic environment,*

Yet, what amounts to the same thing, in order

(2.11) to *relativize* something (: "*covariantly*" detecting/studying it, see also (1.17)), it is *most appropriate* to *find the* suitable *sheaf-theoretic context* within which one should *reformulate the issue* under discussion.

Now, as we shall presently explain, the aforementioned *sheaf-theoretic context* admits further the appropriate *dynamical dressing*: This, together with the *localization sensibility/response* due to the very definitions (: *sheaf theory*) of the framework at issue, yields here, the appropriate terrain pertaining to *quantum relativistic* problems (viz. to such ones in *quantum field theory*). In this connection, see also *A. Mallios* [21, (2.14.1)].

All in all, by summarizing the preceding we can say that

(2.12) up to these days, our *analysis*/"*arithmetic*" has been fundamentally *spatial*, viz. always based on the notion of "*space*" (: euclidean-Cartesian, finite dimensional or not). However, grounded on our nowadays experience, the same should be in effect "*algebraic*"/relational (: "*space*" is anyway fundamentally "*combination of relations*" (*Leibniz*), see also *A. Mallios* [21, 22]).

In this connection we should also remark that

(2.13) all *the aforesaid relations* have been/are being already/always organized into *algebraic structures*, while the same in view of the preceding should further be appropriately *sheafified* in order to respond to *quantum relativistic* questions.

On the other hand, the previous situation would really fall short of the pertinent manner of confronting with problems of *quantum field theory* without the aforementioned *dynamical equipment*; however, this too by virtue of what has been said before has to be "*space independent*", that is again *relational/categorical*. Now, as we shall see presently below, this is exactly the case when working within the *framework of* ADG (: *Abstract (Modern) Differential Geometry*; see, for instance, *A. Mallios* [15, 20], due exactly to the *functorial character* of the same mechanism of ADG (see below).

3 Functional Dynamics

We proceed in this Section to remind briefly the way one can endow a given situation capable of providing a *relativistic* (: *varying*/e.g. *sheaf-theoretic*) *localization* with an appropriate "*dynamical dressing*", as that one hinted at in that foregoing. Therefore, one thus would finally afford a

(3.1) *dynamical relativistic localization*,

within which we would then very likely be able to formulate/solve *quantum field theory* problems, hence such of *quantum relativity*, as well.

Thus, our aim by the following discussion is to point out, in effect to remind (a full account thereon can be found in *A. Mallios*, ibid.), the manner one can afford a fully-fledged *differential-geometric machinery* of the sort we have it in the classical theory (CDG): however now one is based on suitably chosen hypotheses, as the case might be, pertaining to the *type of* our "*arithmetic*" $\mathcal{A}$ that seems to be *appropriate to our purpose*: So in full contradistinction with what happens classically (CDG), *the aforesaid machinery does not rely on any* "*space*" (Euclidean or otherwise), but *only* on $\mathcal{A}$, as it were, since it is always *we* who actually *measure/observe-detect*, see also e.g. (1.13), (1.17). Within this same vein of ideas we further remark that, (3.2) "*geometrical relations/notions*", previously associated with a preexisted (fixed) smooth manifold (: "*space-time*"), are now emanated*from our own perception* (indeed, how else) *of* "*space*", that is, *exactly from* $\mathcal{A}$. (See also (1.12), or even (1.3); we still note that this is also the case, when applying classical reasoning (cf. for instance analytic geometry, CDG).)

Furthermore, it is still worth remarking at this place that according to (3.2),

(3.3) the same notions, as above, acquire now even a *variational form* ($\mathcal{A}$ is varying, cf. (2.9)), *becoming* thus, at the same time, *relativistic*. Yet, the same framework (: always, $\mathcal{A}$) provides the appropriate "*differential*(-geometric) *machinery*" (see below), the latter being thus *varying*, as well. So we are finally led to a situation as described by (3.1).

Note that the above indicates also a *rejection of the* classically (CDG) *fixed background manifold structure*.

We come below to a *brief description of the differential-geometric mechanism* that can be established within the present context, by a *suitably chosen* $\mathcal{A}$ and *without any resort to any sort of* "*space*", in the standard sense of this term; before this, and in anticipation of our subsequent discussion, it would be at least instructive to point out that a *fundamental spin-off* of the aforesaid setup is the following important fact: Namely,

(3.4) one can get *within the point of view of* (3.1) *fundamental differential equations* of the classical theory, as for instance *Einstein's equation* in vacuo as well as, more generally *Yang-Mills equations*; therefore (ibid., see also e.g. *A. Mallios* [21, (2.14)]), even the *quantized form of the same equations* too.

So we come next to remind *basic ingredients of the dynamics* supplied *by ADG*: The same is of an entirely *categorical* (: functional) *character*, not actually requiring any supporting space in the classical sense of the latter term, functioning on objects of a quite general nature (see below). So we work, by assumption, within the category

$$\mathcal{A} - Mod_X \tag{3.5}$$

whose objects are $\mathcal{A}$-*modules* on an arbitrary, in principle, *topological space* X, with $\mathcal{A}$ a *sheaf of algebras* on X whose sections are *unital commutative* (linear associative) *algebras* over the complexes $\mathbb{C}$ (: constant sheaf). In particular, we shall mainly concerned in the sequel with a certain (full) *subcategory of* (3.5) whose objects are *vector sheaves* (: locally free $\mathcal{A}$-modules) on X, *of finite rank*, denoted by

$$\mathcal{V}ect\mathcal{S}h_X^f. \tag{3.6}$$

Thus, our first task is to indicate the (manner, the) *dynamics* (: mechanism of a differential-geometric type), we want to define, acts on the *fundamental issues, which* according to our basic assumption (: "*algebra first*", see e.g. *A. Mallios* [21]) are used to *describe everything we are coped with*; therefore, on our initial *algebra* (sheaf) $\mathcal{A}$ on X, as well. Actually, *everything is* (*locally*) *reduced to* $\mathcal{A}$, that is as we shall see according to the very definitions, *on* the (local) *sections of* $\mathcal{A}$ (cf. for instance (3.10) in the sequel).

So our aim here is first to concoct an appropriate "*reception-space*" to accommodate the transformed (: dynamically changed) elements/sections of $\mathcal{A}$ (: every element of $\mathcal{A}$ can actually be effectuated through a section of $\mathcal{A}$, "*a sheaf is that one of the germs of its sections*"), under the action of the law/process that expresses this *change*/variation; then also *formally determine* the aforementioned law (: "*derivative*"). We are thus tempted to employ a classical device already originated with *Leibniz-Grassmann-Kähler* that was also "geometrically" realized of course by *Newton*, as well. In nowadays parlance, it constitutes an aspect of the "*extension of scalars*" *functor* when speaking algebraically-categorically. Note that our *algebra-sheaf* $\mathcal{A}$ stands here for the "*sheaf/domain of coefficients*", alias, "*generalized scalars*". True, the possibility of working that way is certainly secured (*Kähler*) for any *unital commutative algebra* over the reals $\mathbb{R}$ or the complexes $\mathbb{C}$, as before. This is also *what we actually do*, in the framework of the present *sheaf-theoretic setup*, by appropriately *sheafified* the previous procedure within the category (3.5), or in particular (3.6), by further *axiomatizing the* corresponding *stages* of the same procedure. The latter are still motivated by the *classical theory* (CDG) which *is* thus *always incorporated in the present* extended *context*.

For convenience we recall the first step/hypothesis on the aforesaid axiomatization:thus within the context of (3.5) we assume that we are given the following basic ("*flat*") $\mathcal{A}$-*connection*

$$\partial : \mathcal{A} \longrightarrow \Omega^1, \tag{3.7}$$

satisfying the usual "*Leibniz conditions*" of a classical derivative with Ω^1 denoting the corresponding herewith $\mathcal{A}$-*module* of our "*differential 1-forms*" of the classical theory. Its existence is here, for generality's sake, axiomatically asserted while it can always be achieved, as previously explained, by following *Kähler's device* (: *extension of scalars functor*), being the "*universal derivative*" space (see also below) associated with the "*universal object*" Ω^1. (In this context, we further note that, as a result of the aforesaid "*universality*" of Ω^1, there is a natural map between the "*module of differentials*" of the classical theory and that one defined à la *Kähler*; this can be realized/identified by still taking, for instance, into account the *topological algebra structure* of the *sheaf of smooth functions* in the classical set-up. In this respect, see also e.g. relevant remarks of *D. Eisenbud* [6, p. 389]). Following further the classical device (CDG), always axiomatically proceeding, concerning the individual "*differential operators*", we can arrive for suitable $\mathcal{A}$ to a *de Rham complex*, employing sheaf cohomology whose *exactness is* still *axiomatically assumed.*

The fundamental notion of an $\mathcal{A}$-*connection* referring to an object of (3.5), say $\mathcal{E}$, is now a *functor-morphism* in the *category of* $\mathbb{C}$-*vector space–sheaves* on X denoted by

$$\mathbb{C} - \mathcal{V}ect\mathcal{S}h_X; \tag{3.8}$$

that is, one has the $\mathbb{C}$-*linear morphism*,

$$D : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \equiv \Omega^1(\mathcal{E}) \tag{3.9}$$

that still obeys the *Leibniz condition* expressed *section-wise*, by the relation

$$D(\alpha \cdot s) = \alpha \cdot D(s) + s \otimes \partial(\alpha), \tag{3.10}$$

for any $\alpha \in \mathcal{A}(U)$ and $s \in \mathcal{E}(U)$. Of course, D can also be construed as a*natural transformation of* the *functors* (: complete presheaves $\equiv$ sheaves) appeared in (3.9). Thus, in other words, the so-called $\mathcal{A}$-*connection* as before is in effect our ability of providing *a means to effectuate a* (tested/phenomenal) *variation of the elements of* $\mathcal{A}$, whose *we only know the* ("*formal*") *way they* (: the variations) *behave* (*Leibniz*), so that we then concoct/construct, "*formally*" (again), their world (: "*space of living*":) Ω^1, an $\mathcal{A}$-*module*.

Now, by following classical arguments in Homological Algebra extending/motivated by our standard experience/practice in CDG, one can further develop the corresponding herewith "*differential-geometric*" machinery pertaining thus to the present *abstract framework* (: ADG); this has been already presented, in extenso and every detail in *A. Mallios* [15, 20]. Of course, the above depends on the particular choice of the "*arithmetic*" $\mathcal{A}$, see for instance *A. Mallios–E.E. Rosinger* [24, 25], or even *A. Mallios* [15]. So to supplement our information, provided by (3.9) and thus better also elucidate (3.10), we have to mention that we are given a $\mathbb{C}$-(linear) *morphism* (see also (3.7)),

$$\partial : \mathcal{A} \longrightarrow \Omega^1, \tag{3.11}$$

(: "*flat*" $\mathcal{A}$-connection) satisfying the analogous (familiar) here "*Leibniz condition*", for any product, $\alpha \cdot \beta$, with α, β in $\mathcal{A}(U)$.

On the other hand, a *fundamental moral* based in effect on the classical theory (CDG) is that:

(3.12)

> any "*differential-geometric*" notion, that is needed at any particular stage of developing the machinery, at issue, should be given always *in terms of* $\mathcal{A}$ (: "*sheaf of coefficients*", viz. $\mathcal{A} \equiv$ "*we*", as it were in effect), and in principle via *sheaf morphisms*. It has thus always to be borne in mind here that, by assumption
>
> (3.12.1) *no "spatial" support/input is available*,
>
> something, that is actually the case in reality, as well; cf., for instance, *A. Einstein* [5], in that:
>
> (3.12.2) "*time and space are modes by which we think, not conditions in which we live*".
>
> See e.g. *Yu.I. Manin* [28, p. 71], or even,
>
> (3.12.3) "… *continuous space-time*… *should be banned* from theory as a*supplementary construction* not justified by the essence of the problem—a construction *which corresponds to nothing real*.
>
> Cf. for example *J. Stachel* [37, p. 280].

[Emphasis above is ours]. Besides as already said we also remark, within the same context, that;

(3.13) *everything* here *is functorial*/categorical, *being sheaf-theoretic*.

Furthermore, what is also of a *particular importance* is the fact that,

(3.14) *everything should be*, and *actually is referred to* $\mathcal{A}$ (: *we*), *globally and locally.*

Concerning the latter claim as above, that is "*locally to* $\mathcal{A}$",*this is* actually *always the case* according to the very hypothesis of the same objects/notions involved in ADG (loc. cit.). Furthermore one succeeds in that way to get

(3.15) *everything* (localizable/detectable) *functorial with respect to* $\mathcal{A}$,

a fact of *fundamental significance* for the whole stage of the mechanism of ADG and of its various applications (see e.g. Sect. 4 below); one can further construed the same as the analogue herewith of the classical *principle of general covariance* or even of the *principle of general relativity* by simply taking into account (3.14). See also the applications referred to in the next sections.

4 A Topos-Theoretic Variation

It is admittedly true that,

(4.1) "*. . . the very notion of sheaf is . . . central to topos theory.*"

See, for instance, *S. MacLane–I. Moerdijk* [13, p. 2]. As a result, *the* previous *sheaf-theoretic framework of* ADG is therefore (and can indeed be) quite appropriate to adopt a *topos-theoretic* re*formulation* provided we are still able to transfer to this *more abstract setting*, hence to afford also the "*dynamical part*" of ADG. Note that the *functorial*/categorical *character of the* whole (abstract) "*differential-geometric" machinery of* ADG *is* just *of paramount importance* (: a *fundamental guide*) *to* the accomplishment of *this task*: So we first remark that

(4.2) it is a *basic assumption*, throughout the present discussion, as well as, of the whole perspective of "*Abstract* (or else "*Modern*") *Differential Geometry*" (: ADG), that,

(4.2.1) *the world around us is simply the result of certain particular relations* (: *physical laws*),

to paraphrase, or even post-anticipate herewith *Leibniz*; see also, for instance, *A. Mallios* [21, (2.6)], or [16, (1.1), (1.4], together with [19, (1.1)]. The same aspect, as in (4.2.1) still indicates the *fundamental notion* of a "*field*"; that is of a pair,

$$(\mathcal{E}, D), \tag{4.3}$$

alias, of a "*Yang–Mills field*", in the terminology we adopt throughout ADG (cf. *A. Mallios* [15, 20]). Therefore, that one also of an "*observable*" (: "*field strength*"); in this context, see e.g. *A. Mallios* [21, (2.9)], *E. Zafiris* [41], along with (3.9) and (3.10) in the preceding.

On the other hand, we further remark that it is very convenient (still from a *pedagogical perspective*) to look at a given

(4.4) *presheaf*, as something of an appropriately organized "*information*", while at the corresponding "*complete*" presheaf, in the sense of *Leray*, viz. its associated *sheaf*, as the looked for "*full information*". So this can be supplied or, at least, could be checked up on the basis of suitably (locally) defined "*equivalent information*"; such an "*equivalence*" is here provided, according to the initially afforded "*data of information*", that is via the given presheaf.

The previous procedure of "*completing a* (locally) *given information*", entailing thus the way one can *have a global information* (: sections of a sheaf), *by locally determining it* (: *germs* of sections, *presheaf*), can be related, of course, with the famous/familiar from topos theory "*plus construction*". The latter operation is thus a landing off of the former: Indeed, the aforesaid process constitutes virtually the *conversion of a*(n arbitrary) *presheaf* into a "*localizable" one*, viz. into a "*functional*", so to say, *presheaf*, therefore, more *flexible*; thus, the whole enterprise becomes again a "*functionalization" of* given *arbitrary* (abstract, "*rigid*") *data*, hence, a conversion of the same into such of a (more naturally) localizable type. At the same time, this can become susceptible of being, even *globally perceivable*, and yet all this, in a much more general manner as it is a *topos-theoretic framework*.

The above represent the general idea of "*sheafification*" *of* given *data* (: (local, always) "*information*"), either by referring to a standard arbitrary, in principle, topological space (: classical case), or even to a *site* (viz. *to a Grothendieck topology* and the like, cf. below). So in the second much more general case, a classical topology is replaced by the elements/objects of a given ("*small*") *category*, while a classical open covering by a "*sieve*" over an object of the small category at issue, where on the latter one can then define a *Grothendieck topology*. Thus, one affords in principle, a much greater possibility of dealing with supplied *information* which in the case, for instance, of relevant potential physical applications, e.g. *quantum theory*, may be associated with what one might call "*observables*". All the previous framework is still by the same definitions of a "*varying* (: relativistic) *localizational*" character (sheaf/topos theory), modulo always *the* relevant *dynamics* of the whole edifice (hence, confronting thus with *quantum field theory* as well, see (3.1)).

4.1 Functorial (Topos-Theoretic) Dynamics

Our aim in the following discussion is to provide evidence of the manner that one can afford a

(4.5) "*formal description/expression*" of the way our *basic information*, as this is by assumption represented/modelled by the $\mathbb{C}$-*algebra* (*sheaf*) *of "coefficients*" $\mathcal{A}$, or even by any appropriate $\mathcal{A}$-*module* $\mathcal{E}$, *is varying*. In that context, one simply follows the classical/standard *axiomatization* of the subject at issue, as given by *Leibniz* (being, of course, independently, *vindicated*, by *Newton*): So this is what we formally call herewith an ($\mathcal{A}$-)*connection* (: *"dynamics*") in both a *sheaf* and *topos–theoretic framework*.

So to this end, one has first to concoct the "*space*" *on which the* aforesaid *variations live*, viz. in other words to *define*, "*formally*" again their world, say, Ω^1, thus basically another $\mathcal{A}$-*module*. Equivalently, we summarize the above by saying that

(4.6) to afford an $\mathcal{A}$-*connection* (: "*dynamics*") simply means that we are in the position to *participate the variation of objects* we are interested in, *knowing their domicile*, viz. the $\mathcal{A}$-*module* Ω^1 *together with its relation*(-functor) *with* $\mathcal{A}$, see e.g. (3.11).

Yet, concerning the above procedure/function, one should remark that we are already facilitated/prepared by our *abstract experience* from ADG to cope with it, the same function, as in (4.6) being entirely of a quite *functorial nature*, so that one can further look trustingly after its *topos-theoretic adaptation*:

To start with, suppose we are given a *category* $\mathcal{E}$ together with a *small full separating* (alias, *generating*) subcategory $\mathcal{A}$. We denote this, by

$$\mathcal{A} \underset{\rightarrow}{\subset} \mathcal{E}. \tag{4.7}$$

In this connection, we further note that, by assumption, the previous "*inclusion functor*" (4.7) is still *full* and *faithful*, so that the corresponding *maps between* "Hom-sets" of the above two categories are, in effect, *bijections*. Hence, *we know* $\mathcal{A}$ *by just knowing* (the set of) *its objects* (: arrows/"information" are the same, as in $\mathcal{E}$); see also *S. MacLane* [12, p. 15]. On the other hand, since $\mathcal{A}$ is by hypothesis a *separating* subcategory of $\mathcal{E}$, one concludes based on the preceding terminology that (ibid., p. 123);

(4.8) *parallel arrows in $\mathcal{E}$ can be discerned through arrows in $\mathcal{A}$.*

More technically speaking this means that (see also [13, p. 576]),

(4.9) arrows in $\mathcal{A}$ ending at the same object of $\mathcal{E}$ constitute, as we say, an "*epimorphic family*".

Thus, one can then consider (*Giraud's Theorem*, see e.g. *S. MacLane–I. Moerdijk* [13, p. 580f]) the so-called *Grothendieck topology*, say, J, that can be associated with such (epimorphic) families, as before, getting a *site*

$$(\mathcal{A}, J) \tag{4.10}$$

along with the *category of sheaves on it*, in the sense of *topos-theory*, denoted in the sequel by

$$\mathcal{S}h(\mathcal{A}, J). \tag{4.11}$$

Our aim now is to show the following *category equivalence*,

$$\mathcal{E} \cong \mathcal{S}h(\mathcal{A}, J), \tag{4.12}$$

so that $\mathcal{E}$ becomes then a *Grothendieck topos*. See also (4.17) below along with the ensuing discussion therein for a *physical meaning of* (4.12).

Now, what we wish further to show is that,

(4.13) *quantizing*, *in a topos-theoretic sense* means, in effect, *sheafifying à la Grothendieck*, so that *quantum relativization* (: a *topos-theoretic quantum field theory*) could be attained by a,

(4.13.1) *dynamical sheafification à la Grothendieck/Mallios*.

In this context, one gets at the necessary *topos-theoretic dynamics* by defining it, following ADG, on the *generating subcategory $\mathcal{A}$ of $\mathcal{E}$*. This, for convenience, can be assumed to be a small category of *"classical arithmetics*", viz., in other words, the objects of which are thus taken to be *unital commutative $\mathbb{C}$-algebras* (of observables). Therefore, we actually consider a *small subcategory* of the category *of all unital commutative $\mathbb{C}$-algebras*. It is by means of the previous small subcategory $\mathcal{A}$ that one endows the whole set-up, as in (4.12), with the "*spark*", we are looking for, *of the* relevant "*dynamics*" *in $\mathcal{E}$*, according to the *prototype of* ADG (and the characteristic examples therein; cf. *Mallios–Rosinger, Mallios–Raptis*, as in the Refs. See also, for instance, *A. Mallios* [19, (4.3), §§4.(a), (b)]):

First, based on the very definitions and appropriate supplementary hypotheses, one concludes that;

(4.14) the *objects of $\mathcal{E}$* are expressed, within a *category equivalence*, as *colimits of* the functor category

$$(\mathcal{S}ets)^{\mathcal{A}^{op}}, \tag{4.14.1}$$

that is of the category of *set-valued presheaves on $\mathcal{A}$*; thus, one has the *category equivalence*,

$$\mathcal{O}b(\mathcal{E}) \cong colim\,\mathcal{O}b((\mathcal{S}ets)^{\mathcal{A}^{op}}). \tag{4.14.2}$$

In other words,

> *every object of*, $\mathcal{E}$, being a *sheaf on the site* (4.10) (i.e., for the *Grothendieck topology* J), *is represented as a colimit of representable presheaves from* the category (4.14.1). Indeed, one remarks that,

(4.15)

(4.15.1) *every presheaf* (: object of the category (4.14.1)) *is the colimit of representable presheaves*: See e.g. [13, p. 41, Proposition 1 or p. 42, Corollary 3].

The previous considerations are based on the fact that,

(4.16) there is an "*adjunction*" between the *topos* (4.14.1) and the category $\mathcal{E}$ whose objects are, in effect (: modulo a categorical isomorphism), *sheaves* (cf. (4.14.2)) in the same functor category (4.14.1).

Indeed, by an obvious abuse of notation concerning (4.14.2), one gets at the following useful/informative relation,

(4.17)
$$\mathcal{E} = \varinjlim\big((\mathcal{S}ets)^{\mathcal{A}^{op}}\big),$$

modulo of course the aforesaid *isomorphism* (: category equivalence) as in (4.14.2). As already mentioned, the *presheaves* (: objects of (4.14.1)) appeared in the second member of (4.17) are, in effect, *representable* (cf. (4.15.1)) via the "*injection/inclusion functor*" (4.7) that actually entails a ""*Hom–tensor*" *adjunction*", alluded to already in (4.16). In this respect, see also for instance *S. MacLane-I. Moerdijk* [13, p. 580, "converse part of *Giraud's theorem*"]. The above *sheaf-theoretic representation* of the elements *of the* given (abstract) *category* $\mathcal{E}$ supplies also the way to define on $\mathcal{E}$ what one might denominate as, "*Grothendieck dynamics*" *on* $\mathcal{E}$*, based on* principles of *ADG* (cf. (4.13.1)). Indeed one relies here, in view of the aforesaid *adjunction functor* (cf. (4.16)), on the possibility of expressing the *objects of* $\mathcal{E}$ as *colimits of the functor category* (4.14.1) (loc.cit. along with (4.17)).

The above constitute essentially an *abstract version* of relevant recent and detailed work of *E. Zafiris* in various topos-theoretic contexts for the modelling and interpretation of quantum event [40, 42] and quantum observable structures [39, 41, 43] along sheaf–theoretic lines, see also *A. Mallios-E. Zafiris* [27]: The concrete topos–theoretic scheme, developed by *Zafiris* for that purpose, makes essential use of the notion of a categorical *Grothendieck* topology, interpreted physically by means of *generalized localization systems* of quantum event/observable algebras, consisting of epimorphic families of Boolean or general commutative algebraic coverings. In this perspective, it is proved that quantum event/observable algebras can be made isomorphic with structure sheaves of Boolean/commutative coordinatization coefficients for these localization systems. The main implications of this scheme are related with the conclusion that *globally non-commutative quantum structures are understood via functorial families of local Boolean/commutative reference frames* (see, for instance, (4.17)) pasted together along their overlaps. On the other hand, the sheaf-theoretic representation of quantum structures according to the previous lines provides the basis for the development of a topos-theoretic dynamics for quantum algebras from an algebraic sheaf-cohomological point of view [27, 43] based mainly on ideas from ADG.

Analogous considerations, within a topos-theoretic setup, have been independently supplied in recent work of *I. Raptis* [32, 33] pertaining mainly to a "*finitary causal and quantal*" perspective. In this connection, it is still of interest the ongoing current work of *M. Papatriantafillou* [31], referring to a *categorical study of ADG* that has also been employed in the work of *I. Raptis* [33]. On the other hand, a categorical aspect of ADG has been already independently considered in the work of *E. Zafiris* (ibid.).

Thus, as another result of our considerations in (4.14) and (4.17) one comes to the conclusion that:

(4.18) our description of "*quantum situations*" may be presented in terms of non-commutative issues (e.g., *Heisenberg's "matrix mechanics*"), however, our *calculations are* always *conducted, by* means of *commutative elements* appertaining to *commutative environments* (e.g.*"algebras*").

The latter might still be compared with the classical *"Bohr's correspondence principle*". Therefore, the above together with (4.14.2) leads us to the aspect that the aforesaid relation might still be conceived as a *topos–theoretic interpretation of* "*Bohr's correspondence principle*". However, see also relevant remarks in (1.18) in the preceding, as it concerns, in that context, the general viewpoint of ADG.

Scholium 4.1 In connection with the preceding account and in view of the general perspective dominating ADG, it is also instructive to make the following remarks: So we note here and again, that

(4.19) *the intervention of a topological space in the sheaf-theoretic formulation of* ADG *is* to be considered just as quite *misleading*.

Indeed, the contribution of "*space*" therein, as a support of the sheaves involved, is only in "*parameterizing*", so to say, *the organization of the information encoded*, through the structure of the same sheaves: hence, this, by *not affecting* at all *the intrinsic spaceless character of* ADG, in particular, as this concerns potential applications of the same theory (cf., for instance, *quantum gravity*). As a matter of fact, as noted already throughout the exposition of the theory, the "*space*", in the classical sense of this term, is just the *spin-off of* the "*structural algebra-sheaves*" involved at each particular moment (in this context, see also the next section, along e.g. with *A. Mallios* [21, Sect. 5], or even *A. Mallios* [20, Vol. II; Chap. I, Sect. 7]).

Now, a very characteristic/vindicating example of the above is, in effect, the aforementioned relevant work of *E. Zafiris* (see Refs.), in formulating the same theory (ADG), within a *topos-theoretic* framework, "*suited to the quantum regime*" (cf. [43, Sect. 8, p. 350ff]); here an appropriate *Grothendieck topology* is employed instead in place of a "*measurement topological space*", as in the standard case of a sheaf, indicating thus the *functorial character of the esoteric mechanism*/function *of* ADG. Of course, the same aspect is supplied within the previous point of view, still, by the above axiomatic approach.

Yet, within a similar perspective as before, one may further consider the aforesaid already analogous recent *topos-theoretic treatment* of *I. Raptis* (see Refs.) referring to *quantum gravity* problems, as well; its "*dynamical*" part is still rooted on ideas from ADG.

5 Schemes and ADG

In this final Section of the present treatise we want to highlight a certain potential intermingling of elements of *scheme theory* with fundamental aspects of ADG, as related to *topological algebra theory*, cf. *A. Mallios* [14]. The same has been already exemplified in connection with problems pertaining to a *gauge-theoretic* treatment of *quantum gravity*; see, for instance, *A. Mallios* [20, Vol. II], or even [21, (5.5)–(5.9)]. So the pertinent notion here that seems to fit in quite well with the aforesaid framework is that one of a *topological algebra scheme*: [Having to do with the historical part of the present subject matter, I could say that, in principle, I was always interested in that notion along with potential applications in *topological algebra theory*: this due mainly, in particular, to my own interest in the latter theory, yet in its *sheaf-theoretic perspective*. So it was actually a pleasant instance asked once by *I. Raptis* when, in connection with our joint work on *quantum gravity within the* ADG *setup*, about the above notion. Indeed, that same idea was, in effect, implicitly used already in *A. Mallios* [20, Vol. II; Chap. IV], referring to "*General Relativity as a Gauge Theory*"; the same aspect has been explicitly mentioned thereafter in *A. Mallios* [21], concerning the real contribution within the latter context of the notion at issue]. For convenience, we give below the fundamentals of the above terminology while for the rudiments of topological algebra theory, we refer for instance to *A. Mallios* [14]. Thus suppose that we are given a *topological algebra space*

$$(E, X), \tag{5.1}$$

such that E is a *topological algebra* (ibid.), assumed to be unital and commutative over the complexes $\mathbb{C}$, while X is the *spectrum* (alias, *Gel'fand space*) of E. The same is by definition the set of (continuous) 1-dimensional representations of E, alias *continuous characters* (: $\mathbb{C}$-algebra morphisms) *of* E viewed as a subset of the "*weak topological dual*" *of* E, E'_s (loc. cit.), hence, a (Hausdorff) topological space. The latter is still denoted (for historical reasons) by $\mathfrak{M}(E)$ (ibid.). Now, applying a *sheaf-theoretic* approach to the standard *Gel'fand* theory in *Banach* algebras, the above data yield X, as the *base space* of a ($\mathbb{C}$)-*algebra sheaf* of E, say $\mathcal{E}$, viz. the so-called *Gel'fand sheaf of* E, see *A. Mallios* [17]. In that manner, by analogy with what happens in nowadays *Algebraic Geometry*,

(5.2) one can *associate with a* given *topological algebra* E *a topological space* X *together with an algebra sheaf on it*, say $\mathcal{E}$; the resulting pair

$$(\mathcal{E}, X) \tag{5.2.1}$$

is called a *sheaf* (alias, *affine*) *topological algebra space* associated with E.

The term "*affine*", as before, will be made clear right below. Thus, it may happen that for a suitable topological algebra E, as in (5.1), the latter is (isomorphic with) that one of the global sections of its Gel'fand sheaf $\mathcal{E}$ (alias, "*structure sheaf*" of the pair (5.2.1)), viz. one has

$$E \cong \Gamma(X, \mathcal{E}) \equiv \mathcal{E}(X), \tag{5.3}$$

within a ($\mathbb{C}$-)*algebra isomorphism*. In that case we call E (for obvious reasons), a *geometric topological algebra*. Now, one defines an

(5.4) *affine topological algebra scheme*, as a pair *"of type* (5.2.1)", modulo a suitable categorical isomorphism, that is associated with a geometric topological algebra E.

On the other hand, one naturally defines as a

(5.5) *topological algebra scheme*, any pair as in (5.2.1), modulo an isomorphism (cf. (5.4)), the same pair being *locally* (isomorphic to) *an affine topological algebra scheme*.

We have thus here through the particular case in hand, another realization of the standard fact as it were, for that matter that "*schemes are built up from affine schemes*". On the other hand, the advantage to employ *topological algebra schemes* is, according to concrete examples (see *A. Mallios* [20, Vol. II]), the *access*, by the very definition of the same structure *to* an appropriate *topological-algebraic environment, locally*. The latter is still possible to supply the needed, as the case might be, *differential-geometric mechanism* to treat "*dynamical*" issues in the sense of ADG, viz. without any surrogating "*space*" as it actually happens in the latter theory. So one realizes still here the deeper characteristic of the *geometry of schemes* in the sense that *schemes are made from gluing together suitable* "*local pieces*", *the* "*affine schemes*" as above; this in particular within the topological algebras framework considered herewith. Yet, as for the time being, it might be said that the "*geometry of schemes*" (see e.g. [7]), when *in conjunction with* ADG (and, occasionally, with *topological algebra theory* in that context, loc. cit.), therefore, with potential applications in *quantum gravity* too (ibid.), seems to be more flexible/(directly)effective in comparison with the analogous situation one has through the "*geometry*" in terms of *topos-theory*.

Epilogue The *axiomatic approach to* the so-called *quantum gravity* as advocated by the preceding, can virtually be associated, when of course appropriately adjusted, with the entertainment of formulating any (physical) theory, pertaining to the variation of a given (algebraically) "*organized information*". Hence one could also be able to provide ("*differential*") *equations*. Thus, in other words to conduct the necessary "*calculations*" thereat referring to the *laws that* seem to condition the *information* we possess. So we ascertain here what one might call "*relational dynamics*", that is "*dynamics*" without the intervention of any "*space*". Indeed, "*dynamics*" is a matter of the functions/functors that intervene in a given procedure, not of a "*space*". In other words, we are concerned with "dynamics" *referring* directly *to* the *relations* we are interested in. Yet, the "*trick*" herewith is to have the *relations* at issue *grouped algebraically* together, so that one can then think, for instance, of an appropriate *presheaf*, or even a *scheme* (see the preceding). This is still a sort of "*dynamics*" that might be traced back to *Leibniz, Kähler*, or even to *Feynman*, concerning the classical so-called "*diagrams*"-theory of the latter.

On the other hand, regarding the occasional "*singularities*" one might be confronted with when applying a given "*arithmetic*" $\mathcal{A}$, one should then look for another more appropriate one $\mathcal{A}' \supseteq \mathcal{A}$, that keeps the"*mechanism*" of the former "*untouched*" and moreover "*absorbs*" *the* eventual "*singularities*" of $\mathcal{A}$. In that context, a further potential application of ADG seems very likely to be "*Geometric Measure Theory*"; see for example the recent account of *F. Morgan* [29]. Yet, also *A. Mallios* [15, Vol. II], concerning the rôle herewith in connection with ADG, as before, of the so-called *Lipschitz functions* (loc. cit., p. 298); yet, see *F. Morgan* [29, p. 1, along with p. 21, 3.2: *Rademacher's Theorem*]. As a matter of fact, the above can be construed as a very special case of the point of view disseminated already by [24, 25].

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